MULTILINEAR SQUARE FUNCTIONS AND MULTIPLE WEIGHTS

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Abstract

In this paper we prove weighted estimates for a class of smooth multilinear square functions with respect to multilinear $A_{\vec{P}}$ weights. In particular, we establish weighted estimates for the smooth multilinear square functions associated with disjoint cubes of equivalent side-lengths. As a consequence, for this particular class of multilinear square functions, we provide an affirmative answer to a question raised by Benea and Bernicot [2] about unweighted estimates for smooth bilinear square functions.

1. Introduction

Let $\phi(\xi_1, \xi_2, \dots, \xi_m)$ be a bounded measurable function defined on the $m-$fold Cartesian product $\mathbb{R}^n$ with $m \geq 2, n \geq 1$. For $m-$tuples of nice functions $\vec{f} = (f_1, f_2, \dots, f_m)$, the multilinear multiplier operator T_ϕ associated with the symbol ϕ is defined by

$$T_\phi(\vec{f})(x) \quad = \quad \int_{\mathbb{R}^{mn}} \phi(\xi_1, \xi_2, \dots, \xi_m) \exp\big(2\pi i (\sum_{j=1}^{m} x \cdot \xi_j)\big) \prod_{j=1}^{m} \hat{f}_j(\xi_j) d\xi_j, \quad (1)$$

where $x \cdot y$ denotes the standard inner product of vectors x and y in $\mathbb{R}^n$.

The above expression for $T_\phi(\vec{f})(x)$ in space variables for functions f_j takes the form

$$T_\phi(\vec{f})(x) \quad = \quad \int_{\mathbb{R}^{mn}} K(y_1, y_2, \dots, y_m) \prod_{j=1}^{m} f_j(x - y_j) dy_j, \quad (2)$$

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where $K = \check{\phi}$, provided the integral is interpreted appropriately as an action of a distribution to a tensor product of functions, if necessary.

The modern theory of multilinear operators is motivated by the remarkable works of Lacey and Thiele [9, 10] on L^p-boundedness of the bilinear Hilbert transform. The bilinear Hilbert transform is a bilinear singular integral operator possessing a crucial property of modulation invariance. There has been a considerable progress in understanding the core issues and difficulties in the area of multilinear singular integral operators in the past decade.

In this paper we are concerned with operators that are closely related to multilinear Calderón-Zygmund operators. The multilinear Calderón-Zygmund theory has been studied and developed systematically by Grafakos and Torres [7]. The weighted theory of these operators has been developed by Lerner et al [13]; in this paper the authors study a maximal function which plays the analogous role in the multilinear Calderón-Zygmund theory as the classical Hardy-Littlewood maximal function plays in the context of the linear Calderón-Zygmund operators. They also introduce multilinear $A_{\vec{P}}$ weights which completely characterize the weighted L^p-boundedness of this multilinear maximal operator.

Given an $m-$tuple of locally intergrable functions $\vec{f} = (f_1, f_2, \dots, f_m)$, the multilinear maximal operator $\mathcal{M}(\vec{f}\,)$ is defined by

$$\mathcal{M}(\vec{f}\,)(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \frac{1}{|Q|} \int_Q |f_j(y_j)| dy_j, \ x \in \mathbb{R}^n.$$

For an $m-$tuple of exponents $p_1, p_2, \dots, p_m$ we will denote by p the exponent given by $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\vec{P} = (p_1, \dots, p_m)$.

Definition 1.1. *Let $1 \leqslant p_1, \dots, p_m < \infty$. Given an $m-$tuple of weights $\vec{w} = (w_1, w_2, \dots, w_m)$, set*

$$v_{\vec{w}} = \prod_{j=1}^{m} w_j^{p/p_j}.$$

We say that $\vec{w}$ satisfies the multilinear $A_{\vec{P}}$ condition if, and only if

$$\left(\frac{1}{|Q|}\int_Q v_{\vec{w}} dy\right)^{1/p} \prod_{j=1}^{m} \left(\frac{1}{|Q|}\int_Q w_j^{1-p_j'} dy_j\right)^{1/p_j'} \leqslant K \tag{3}$$

for all cubes Q.

Here we follow the standard interpretation of the average $\left(\frac{1}{|Q|}\int_Q w^{1-p_j'} dy_j\right)^{\frac{1}{p_j'}}$ as $(\operatorname{ess\,inf}_Q w_j)^{-1}$ when $p_j = 1$.

Observe that the Muckenhoupt's classical A_p weights always give rise to multilinear $A_{\vec{P}}$ weights in the sense that

$$\prod_{j=1}^{m} A_{p_j} \subset A_{\vec{P}}.$$

Moreover the preceding inclusion relation is strict. See [13] for more details and important properties of multilinear $A_{\vec{P}}$ weights.

The following characterization of the multilinear $A_{\vec{P}}$ weights in terms of the classical A_p weights has been proved in [13]. This will be used in the proof of our main result.

THEOREM 1.2. [13] *Let $\vec{w} = (w_1, w_2, \ldots, w_m)$ and $1 \leqslant p_1, \ldots, p_m < \infty$. Then $\vec{w} \in A_{\vec{P}}$ if, and only if*

$$\begin{aligned} &w_j^{1-p_j'} \in A_{mp_j'}, \quad j = 1, 2, \ldots, m \text{ and} \\ &v_{\vec{w}} \in A_{mp}, \end{aligned}$$

where the condition $w_j^{1-p_j'} \in A_{mp_j'}$ is understood as $w_j^{1/m} \in A_1$ when $p_j = 1$.

The multilinear $A_{\vec{P}}$ weights completely characterize weighted L^p- boundedness of multilinear maximal function $\mathcal{M}(\vec{f})$ in the following sense.

THEOREM 1.3. [13] *Let $1 < p_j < \infty, j = 1, 2, \ldots, m$, be such that $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Then the strong-type weighted inequality*

$$\|\mathcal{M}(\vec{f})\|_{L^p(v_{\vec{w}})} \quad \lesssim \quad \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)} \tag{4}$$

holds for every $\vec{f} = (f_1, f_2, \ldots, f_m)$ if, and only if $\vec{w} = (w_1, w_2, \ldots, w_m)$ satisfies the multilinear $A_{\vec{P}}$ condition.

We refer to [4, 11, 12] for recent developments on weighted estimates for multilinear operators using sparse domination principle.

Throughout this paper, the notation $A \lesssim B$ is used to indicate that there is a constant $C > 0$ such that $A \leq CB$.

2. Multilinear square functions

Let $\{Q\}_{Q \in \Omega}$ be a collection of disjoint cubes in $\mathbb{R}^{mn}$. Let Φ_Q be smooth functions adapted[1] to cubes $Q \in \Omega$. Let $T_{\Phi_Q}(\vec{f})$ denote the multilinear multiplier operator

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[1] this means that $\Phi_Q(x) = \Phi(\frac{x - c_Q}{\ell(Q)})$, where c_Q is the center of Q, $\ell(Q)$ is its length, and Φ is supported in the double of Q and equals 1 on Q.

associated with the function Φ_Q. The multilinear smooth square function $T_\Omega(\vec{f})$ associated with the collection Ω is defined by

$$T_\Omega(\vec{f})(x) := \left(\sum_{Q\in\Omega} |T_{\Phi_Q}(\vec{f})(x)|^2 \right)^{\frac{1}{2}}. \tag{5}$$

We would like to remark that obtaining L^p-estimates for multilinear square functions is much harder than its classical counterpart. In his celebrated paper [17], Rubio de Francia proved L^p-estimates for the classical Littlewood-Paley operator associated with an arbitrary sequence of disjoint intervals in $\mathbb{R}$. An analogue of this result in the multilinear setting is an open problem to this day. In the theory of classical Fourier multipliers, by the virtue of the Plancherel theorem, the L^2-estimates for the Littlewood-Paley operators always hold. However, in the multilinear case there is no preferred L^p-space for which we a priori have boundedness of the multilinear operators under consideration. Furthermore, the multiplier symbols for the multilinear operators under consideration have their supports in higher dimensional spaces and the geometry of disjoint cubes pose additional difficulties in higher dimensions in dealing with such operators.

The main motivation of this paper comes from a recent work of Benea and Bernicot [2] on the bilinear square functions. In their paper the authors addressed the bilinear case (i.e. $m = 2$) and considered the ℓ_r-valued operator

$$T_{\Omega,r}(\vec{f})(x) := \left(\sum_{Q\in\Omega} |T_{\Phi_Q}(\vec{f})(x)|^r \right)^{\frac{1}{r}}, \ r > 2 \tag{6}$$

and proved the following.

Theorem 2.1. [2] *Let $\{Q\}_{Q\in\Omega}$ be a collection of disjoint cubes in $\mathbb{R}^2$ and Φ_Q be smooth bump functions adapted to cube $Q \in \Omega$. If $r' < p_1, p_2 \leq \infty$ and $\frac{r'}{2} < p < r$ are such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, then we have*

$$\|T_{\Omega,r}(\vec{f})\|_p \lesssim \|f_1\|_{p_1} \|f_2\|_{p_2}. \tag{7}$$

The proof of this theorem in very involved and relies on sophisticated techniques from time-frequency analysis. The authors developed suitable time-frequency techniques to deal with the geometry of cubes. The use of l_r-norm is due to the method of their proof and does not seem to address the question of bilinear square function (i.e., $r = 2$).

We emphasize that Theorem 2.1 establishes un-weighted L^p-estimates for ℓ_r-valued, $r > 2$, bilinear operators on $\mathbb{R} \times \mathbb{R}$. The question for bilinear square functions, i.e., $r = 2$, remains open. In this paper we provide an affirmative answer to this question for smooth square function associated with collection of disjoint cubes whose side lengths are equivalent. We not only address the question in the general setting of multilinear smooth square functions defined in $\mathbb{R}^n$, but also prove weighted estimates at the same time for the best possible range of exponents and multilinear $A_{\vec{P}}$ weights. This gives us a complete analogue of the corresponding classical result about Littlewood-Paley operators in the context of multilinear multiplier operators. The proof of our main result (Theorem 3.1) is motivated from the ideas presented in [15] (see also [6]) for the bilinear square functions and [16] for the classical Littlewood-Paley operators.

3. Main Result

We prove the following result.

THEOREM 3.1. *Let $\{Q\}_{Q\in\Omega}$ be a collection of disjoint cubes in $\mathbb{R}^{mn}$ and Φ_Q be smooth bump functions adapted to the cube Q such that*

$$\sup_Q \int_Q |(1-\Delta)^N \Phi_Q(\xi)|^2 d\xi = C < \infty, \tag{8}$$

where $N > mn\big(\frac{1}{4} + \frac{1}{2}\max(\frac{1}{p}, 1)\big)$. Let $p_1, p_2, \ldots, p_m \geq 2$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = \frac{1}{p}$. Then the weighted inequality

$$\|T_\Omega(\vec{f})\|_{L^p(v_{\vec{w}})} \lesssim \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)} \tag{9}$$

holds for all $\vec{f}$ and every $\vec{w} = (w_1, w_2, \ldots, w_m) \in A_{\frac{\vec{P}}{2}}$, where $\frac{\vec{P}}{2} = (\frac{p_1}{2}, \frac{p_2}{2}, \ldots, \frac{p_m}{2})$.

REMARK 3.2. Recall that in the classical setting the Littlewood-Paley operator associated with an arbitrary sequence of disjoint intervals is bounded from $L^p(w)$ into itself for $2 < p < \infty$ if $w \in A_{\frac{p}{2}}$. See [17] for details. Therefore, Theorem 3.1 provides a complete multilinear analogue of the corresponding classical result.

Proof. We shall use the notation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{Z}^{mn}$, where each of $\alpha_j, j = 1, 2, \ldots, m$, lies in $\mathbb{Z}^n$. We also set $|\alpha| = \sum_{j=1}^m |\alpha_j|$, where $|\alpha_j|$ is the the sum of the entries of α_j.

We prove that the multilinear square function $T_\Omega(\vec{f})$ is dominated, in the pointwise a.e. sense, by a suitable form involving multilinear averages. More precisely, we show that there exist cubes R_α centered at 0 such that

$$|T_\Omega(\vec{f})(x)| \lesssim \sum_{\alpha\in\mathbb{Z}^{mn}} \frac{|\alpha|^{mn/2}}{(1+|\alpha|)^{2N}} \left(\prod_{j=1}^{m} \frac{1}{|R_\alpha|} \int_{x+R_\alpha} |f_j(y_j)|^2 \, dy_j\right)^{\frac{1}{2}} \text{ for } a.e.\ x \tag{10}$$

where N is as in the hypothesis of the theorem.

Let $\{a_Q\}$ be a square summable sequence of scalars with $\sum_Q |a_Q|^2 = 1$. For each $\alpha \in \mathbb{Z}^{mn}$, let Q_α denote the cube in $\mathbb{R}^{mn}$ obtained by translating the unit cube in $\mathbb{R}^{mn}$ by α. Note that the cube Q_α may be written as the product of cubes in $\mathbb{R}^n$

$$Q_\alpha = \prod_{j=1}^{m} Q_{\alpha_j}.$$

Setting $\Psi(y_1, y_2, \ldots, y_m) = \sum_{Q\in\Omega} a_Q \check{\Phi}_Q(y_1, y_2, \ldots, y_m)$ we write

$$\begin{aligned}
\left|\sum_{Q\in\Omega} a_Q T_{\Phi_Q}(\vec{f})(x)\right| &= \left|\sum_{Q\in\Omega} a_Q \int_{\mathbb{R}^{mn}} \check{\Phi}_Q(y_1, y_2, \ldots, y_m) \prod_{j=1}^{m} f_j(x-y_j) dy_j\right| \\
&= \left|\int_{\mathbb{R}^{mn}} \sum_{Q\in\Omega} a_Q \check{\Phi}_Q(y_1, y_2, \ldots, y_m) \prod_{j=1}^{m} f_j(x-y_j) dy_j\right| \\
&= \left|\int_{\mathbb{R}^{mn}} \Psi(y_1, y_2, \ldots, y_m) \prod_{j=1}^{m} f_j(x-y_j) dy_j\right| \\
&\leq \sum_{\alpha\in\mathbb{Z}^{mn}} \int_{Q_\alpha} |\Psi(y_1, y_2, \ldots, y_m)| \prod_{j=1}^{m} |f_j(x-y_j)| dy_j \\
&\leq \sum_{\alpha\in\mathbb{Z}^{mn}} \left(\int_{Q_\alpha} \prod_{j=1}^{m} |f_j(x-y_j)|^2 \, d\vec{y}\right)^{\frac{1}{2}} \qquad (11) \\
&\qquad \left(\int_{Q_\alpha} |\Psi(y_1, y_2, \ldots, y_m)|^2 d\vec{y}\right)^{\frac{1}{2}}.
\end{aligned}$$

Plancherel's identity yields

$$\begin{aligned} \|(1+|\cdot|^2)^N\Psi(.)\|_{L^2(Q_\alpha)} &\leq \left\|(1-\Delta)^N\hat{\Psi}(\cdot)\right\|_2 \\ &= \left\|(1-\Delta)^N\sum_Q a_Q\Phi_Q\right\|_2 \\ &= \left\|\sum_Q a_Q(1-\Delta)^N\Phi_Q\right\|_2. \end{aligned}$$

Since supp $(\Phi_Q)\subseteq Q$ and the collection $\{Q\}_{Q\in\Omega}$ consists of disjoint cubes, we obtain that

$$\|(1+|\cdot|^2)^N\Psi(\cdot)\|^2_{L^2(Q_\alpha)} \leq \sum_Q |a_Q|^2\int_Q |(1-\Delta)^N\Phi_Q|^2 \leq C.$$

Therefore, we obtain

$$\|\Psi\|_{L^2(Q_\alpha)} \lesssim \frac{1}{(1+|\alpha|)^{2N}}.$$

Substituting the above in (11) and using a duality argument for ℓ_2, we obtain

$$|T_\Omega(\vec{f})(x)| \lesssim \sum_{\alpha\in\mathbb{Z}^{mn}} \frac{1}{(1+|\alpha|)^{2N}}\left(\int_{Q_\alpha}\prod_{j=1}^m |f_j(x-y_j)|^2 d\vec{y}\right)^{\frac12},$$

where N is a large positive integer. Now for each $\alpha\in\mathbb{Z}^{mn}$ we set

$$R_\alpha = \Big[-\max_{1\le i\le m,1\le j\le n}|\alpha_{ij}|, \max_{1\le i\le m,1\le j\le n}|\alpha_{ij}|\Big]^n.$$

Then R_α is a cube in $\mathbb{R}^n$ centered at 0 with $|R_\alpha|\le(2|\alpha|)^n$ which satisfies

$$x-Q_\alpha\subset(x+R_\alpha)^m$$

for all $x\in\mathbb{R}^n$. Then we have

$$\left(\int_{Q_\alpha}\prod_{j=1}^m |f_j(x-y_j)|^2 d\vec{y}\right)^{\frac12} \le |R_\alpha|^{\frac{m}{2}}\left(\prod_{j=1}^m\left(\frac{1}{|R_\alpha|}\int_{x+R_\alpha}|f_j(y_j)|^2dy_j\right)\right)^{\frac12}. \quad (12)$$

This yields (10).

In view of the convergence of the sum in (14), it suffices to obtain weighted estimates for each term of the sum in (10) separately. For every j, set $r_j=\frac{p_j}{2}$

and note that $r_j \geq 1$. Hölder's inequality with exponents r_j and r_j' yields

$$\frac{1}{|R_\alpha|}\int_{x+R_\alpha}|f_j(y_j)|^2 dy_j \quad\leq\quad \left(\frac{1}{|R_\alpha|}\int_{x+R_\alpha}|f_j(y_j)|^{p_j} w_j dy_j\right)^{\frac{1}{r_j}} \left(\frac{1}{|R_\alpha|}\int_{x+R_\alpha} w_j^{1-r_j'} dy_j\right)^{\frac{1}{r_j'}}.$$

When $r_j = 1$, the average $\left(\frac{1}{|R_\alpha|}\int_{x+R_\alpha} w_j^{1-r_j'} dy_j\right)^{\frac{1}{r_j'}}$ is interpreted as $(\operatorname{ess\,inf}_{R_\alpha} w_j)^{-1}$. Therefore we obtain,

$$\begin{aligned}
&\prod_{j=1}^m \left(\frac{1}{|R_\alpha|}\int_{x+R_\alpha}|f_j(y_j)|^2 dy_j\right)\\
&\quad\leq \prod_{j=1}^m \left(\frac{1}{|R_\alpha|}\int_{x+R_\alpha}|f_j(y_j)|^{p_j} w_j dy_j\right)^{\frac{1}{r_j}} \left(\frac{1}{|R_\alpha|}\int_{x+R_\alpha} w_j^{1-r_j'} dy_j\right)^{\frac{1}{r_j'}}\\
&\quad\leq \prod_{j=1}^m \left(\frac{1}{v_{\vec{w}}(x+R_\alpha)}\int_{x+R_\alpha}|f_j(y_j)|^{p_j} w_j dy_j\right)^{\frac{1}{r_j}}\\
&\qquad\qquad \left(\frac{v_{\vec{w}}(x+R_\alpha)}{|R_\alpha|}\right)^{\frac{2}{p}} \prod_{j=1}^m \left(\frac{1}{|R_\alpha|}\int_{x+R_\alpha} w_j^{1-r_j'} dy_j\right)^{\frac{1}{r_j'}}\\
&\quad\leq [\vec{w}]_{A_{\vec{P}/2}} \prod_{j=1}^m \left(\mathcal{A}_{R_\alpha, v_{\vec{w}}}\left(|f_j|^{p_j}\frac{w_j}{v_{\vec{w}}}\right)(x)\right)^{\frac{1}{r_j}},
\end{aligned}$$

where $A_{Q,w}(f)$ is the weighted average

$$\mathcal{A}_{Q,w}(f)(x) := \frac{1}{w(x+Q)}\int_{x+Q}|f| w\, dy.$$

It follows that

$$\left(\prod_{j=1}^m \frac{1}{|R_\alpha|}\int_{x+R_\alpha}|f_j(y_j)|^2 dy_j\right)^{\frac{1}{2}} \leq [\vec{w}]_{A_{\vec{P}/2}}^{\frac{1}{2}} \left[\mathcal{A}_{R_\alpha, v_{\vec{w}}}\left(|f_j|^{p_j}\frac{w_j}{v_{\vec{w}}}\right)(x)\right]^{\frac{1}{p_j}}. \tag{13}$$

In view of (13) and (10), we obtain that the multilinear square function $T_\Omega(\vec{f})$ is dominated by a sum of terms involving weighted averages $A_{R_\alpha, v_{\vec{w}}}(|f_j|^{p_j}\frac{w_j}{v_{\vec{w}}})$. First observe that it suffices to prove the desired estimate on the $L^p(v_{\vec{w}})-$norm of a single term $\prod_{j=1}^m \left[\mathcal{A}_{R_\alpha, v_{\vec{w}}}(|f_j|^{p_j}\frac{w_j}{v_{\vec{w}}})\right]^{\frac{1}{p_j}}$, with a uniform constant with respect to the length of R_α. Note that as $p_j \geq 2$ we have $\frac{2}{m} \leq p$. If $p \geq 1$, the assertion follows by Minkowski's inequality and the convergence of the series

$$\sum_{\alpha \in \mathbb{Z}^{mn}} \frac{|\alpha|^{mn/2}}{(1+|\alpha|)^{2N}} \tag{14}$$

for $N > \frac{3}{4}mn$. This condition coincides with the condition in the statement of the theorem when $p \geq 1$. Now if p satisfies $\frac{2}{m} \leq p < 1$, then we use the inequality that $(\sum_\alpha |C_\alpha|)^p \leq \sum_\alpha |C_\alpha|^p$ instead of Minkowski's inequality to deduce the desired estimate provided the series

$$\sum_{\alpha \in \mathbb{Z}^{mn}} \frac{|\alpha|^{pmn/2}}{(1+|\alpha|)^{2Np}}$$

converges; but this is the case since $N > \left(\frac{1}{4} + \frac{1}{2p}\right)mn$ when $p < 1$.

Hölder's inequality yields

$$\left\| \prod_{j=1}^m \left[\mathcal{A}_{R_\alpha, v_{\vec{w}}} \left(|f_j|^{p_j} \frac{w_j}{v_{\vec{w}}} \right) \right]^{\frac{1}{p_j}} \right\|_{L^p(v_{\vec{w}})} \leq \prod_{j=1}^m \left\| \mathcal{A}_{R_\alpha, v_{\vec{w}}} \left(|f_j|^{p_j} \frac{w_j}{v_{\vec{w}}} \right) \right\|_{L^1(v_{\vec{w}})}^{\frac{1}{p_j}}.$$

Furthermore, in order to prove the desired estimate (9), it is enough to show that for doubling weights w, we have

$$\|\mathcal{A}_{Q,w}(f)\|_{L^1(w)} \lesssim \|f\|_{L^1(w)} \tag{15}$$

with bounds independent of $l(Q)$.

Let Q be a cube whose center is at the origin and denote $Q(x) = x + Q$. Then

$$\begin{aligned} A_{Q,w}(f)(x) &= \frac{1}{w(Q(x))} \int_{Q(x)} |f| w dy \\ &= \frac{1}{w(Q(x))} \int_Q |f(x-y)| w(x-y) dy. \end{aligned}$$

Integrating the above with respect to $w(x)dx$,

$$\begin{aligned} \int_{\mathbb{R}^n} A_{Q,w}(f)(x) w(x) dx &= \int_Q \int_{\mathbb{R}^n} \frac{1}{w(Q(x))} |f(x-y)| w(x-y) w(x) dx dy \\ &= \int_Q \int_{\mathbb{R}^n} \frac{1}{w(Q(x+y))} |f(x)| w(x) w(x+y) dx dy \\ &= \int_{\mathbb{R}^n} |f(x)| w(x) \int_Q \frac{1}{w(Q(x+y))} w(x+y) dy dx. \end{aligned}$$

Since w is doubling we get that $w(Q(x+y)) \approx w(Q(x)$ for all $y \in Q$ with constant independent of y. Therefore, the term

$$\int_Q \frac{1}{w(Q(x+y))} w(x+y)dy \lesssim 1.$$

This completes the proof.

REMARK 3.3. We would like to indicate that the pointwise a.e. estimate (10) with the multi-linear maximal function

$$\mathcal{M}_2(\vec{f})^2(x) := \sup_{Q \ni x} \prod_{j=1}^{m} \frac{1}{|Q|} \int_Q |f_j(y_j)|^2 d\mu(y_j),\ x \in \mathbb{R}^n$$

on the right hand side in the inequality (10) can be obtained rather easily.

In a straightforward manner Theorem 1.3 implies that the operator $\mathcal{M}_2$ satisfies the strong-type weighted estimates

$$\|\mathcal{M}_2(\vec{f})\|_{L^p(v_{\vec{w}})} \quad \lesssim \quad \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j)}$$

for $2 < p_j < \infty, j = 1, 2, \ldots, m$, with $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ and every multilinear weight $\vec{w} = (w_1, w_2, \ldots, w_m)$ in the class $A_{\vec{P}/2}$. As a consequence we can obtain the weighted estimates for multilinear square functions for all $2 < p_j < \infty, j = 1, 2, \ldots, m$,

However, this approach does not yield strong-type weighted estimates at end-points, i.e. when $p_j = 2$ for some $j's$. We have used the weighted averaging operators to circumvent this issue and obtained strong type weighted estimates for the entire possible range of exponents in Theorem 3.1.

4. Application to bilinear square function associated with strips

The bilinear multiplier operators associated with symbols of the form $\phi(\xi - \eta)$ are of specific interest and fall under the category of modulation invariant operators. The well-known bilinear Hilbert transform is an important example of such operators. See [4, 9, 10] for more details.

The bilinear Littlewood-Paley operators associated with such bilinear multipliers may be defined in a similar fashion. We refer the interested reader to [1, 3, 5, 6, 8, 14, 15] and the references therein for some relevant background on this.

Here we focus on the following situation. Let $\{I_j\}_{j \in \mathbb{N}}$ be a sequence of intervals with $|I_j| \approx 1$ for all j and let ϕ_j be a smooth function adapted to interval I_j.

The bilinear smooth square function associated with the sequence $\{\phi_j\}_j$ is defined by

$$S(f,g)(x) := \left(\sum_{j\in\mathbb{N}} |S_{\phi_j}(f,g)(x)|^2\right)^{\frac{1}{2}},$$

where S_{ϕ_j} is the bilinear multiplier operator associated with symbol $\phi_j(\xi-\eta)$.

Consider a smooth function ψ supported in $[-1,1]$ such that $\sum_k \psi(\xi-k)=1$. For each j we have

$$\phi_j(\xi-\eta) = \sum_k \phi_j(\xi-\eta)\psi_k(\xi+\eta-k).$$

Denote $\Phi_{j,k}(\xi,\eta)=\phi_j(\xi-\eta)\psi(\xi+\eta-k)$ and note that $\Phi_{j,k}$ are smooth bilinear multipliers whose supports have bounded overlaps.

For compactly supported functions f,g, and h consider

$$\begin{aligned}
\langle S_j(f,g),h\rangle &= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)\phi_j(\xi-\eta)\hat{h}(\xi+\eta)d\xi d\eta\\
&= \sum_k \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)\Phi_{j,k}(\xi,\eta)\chi_{A_{j,k}}(\xi+\eta)\hat{h}(\xi+\eta)d\xi d\eta\\
&= \sum_k \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)\Phi_{j,k}(\xi,\eta)\theta_{j,k}(\xi+\eta)\hat{h}(\xi+\eta)d\xi d\eta,
\end{aligned}$$

where $A_{j,k}=\{\xi+\eta : (\xi,\eta)\in \operatorname{supp}(\Phi_{j,k})\}$ and $\theta_{j,k}$ is a smooth function supported in $2A_{j,k}$ with $\theta_{j,k}=1$ on $A_{j,k}$. Therefore, we have

$$|\langle S_j(f,g),h\rangle| \le \int_{\mathbb{R}} \left(\sum_k |T_{\Phi_{j,k}}(f,g)(x)|^2\right)^{\frac{1}{2}} \left(\sum_k |\tilde{T}_{j,k}(h)(x)|^2\right)^{\frac{1}{2}} dx,$$

where $\tilde{T}_{j,k}(h)=(\theta_{j,k}\hat{h})\check{}$ is the Fourier multiplier operator with smooth symbol $\theta_{j,k}$.

Note that for each fixed j, the intervals $A_{j,k}$ have bounded overlaps and since they are of equivalent size, the supports of $\theta_{j,k}$ also have bounded overlaps uniformly with respect to j. Consequently, for each j the Littlewood-Paley operator associated with the multiplier sequence $\theta_{j,k}$ is bounded on L^p for $p\ge 2$ with a uniform bound with respect to j.

Therefore, we obtain

$$\|S_j(f,g)\|_2 \lesssim \left\|\Big(\sum_k |T_{\Phi_{j,k}}(f,g)(x)|^2\Big)^{\frac{1}{2}}\right\|_2$$

Using the boundedness of the bilinear smooth square function

$$(f,g) \to \left(\sum_{j,k} |T_{\Phi_{j,k}}(f,g)(x)|^2 \right)^{\frac{1}{2}}$$

we obtain

$$\Big\| \Big(\sum_{j} |S_j(f,g)(x)|^2 \Big)^{\frac{1}{2}} \Big\|_2 \lesssim \|f\|_p \|g\|_q$$

for all $p, q > 0$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$.

An analogous problem with rough cutoffs was addressed by Bernicot [1].

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